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Simultaneous Stabilization of Second Order Linear Switched Systems Based on Superstability and D-Decomposition Technique

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Abstract—The considered problem is to simultaneously stabilize a family of second order linear systems by static linear state feedback when applied to switched systems. The proposed synthesis approach is based on a known design method where a static regulator is found as a solution to the linear programming problem. This regulator makes all matrices from the family forming switched systems superstable in the closed loop state, which in turn guarantees exponential stability of the switched system. This approach is generalized for the case where not all matrices in the family can simultaneously be made superstable: for non-superstabilizable matrices one determines using D-decomposition linear bounds on the set of stabilizing regulators, which are used in the linear programming problem. The designed switched system properties are briefly studied. An example of a design problem solution using the proposed approach is presented.

Keywords: non-stationary systems, controller design, superstability, static state regulator, simultaneous stabilization, switched systems

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1. INTRODUCTION

The problem of *simultaneous stabilization* is to find a common control law providing stability for all elements of a certain family of dynamic plants. Such problems form a separate class in robust control theory. They aroused the greatest interest in the period from the late '80s to the '90s of the 20th century [1–6]. The most popular, in practical sense, approach for their solution was proposed in [2, 7] and is based on linear matrix inequalities (LMI), which are related to emergence of effective numerical methods for solving semidefinite programming problems.

Conceptually similar synthesis problems arise in the theory of switched systems [8, 9], which studies non-stationary systems of a special class. Parameters (and structure) of such systems are changing in time according to a certain rule called *switching law* (*or switching signal*). Further, only linear switched systems are considered, where linear subsystems matrices from a certain given set act as the changing parameters (between which switching occurs). The solution to the simultaneous stabilization problem for switched systems only provides a necessary condition for the system's asymptotic stability, which does not take into account the switching effect on the switched system stability. However, in the most general case, the controller design problem for switched systems is to find the regulator that guarantees asymptotic stability of the system for any arbitrary switching law. This design problem is much more complex than just simultaneous stabilization. The main approach for solving it is to look for a regulator that provides the existence of a common Lyapunov function for all subsystems.

The described general problem of linear switched systems stabilization was studied, in particular, in [10, 11], where a new approach to its solution using a static state regulator was proposed. The main idea of the proposed approach was the use of *superstability*, which in relation to control theory was proposed and studied by B.T. Polyak and P.S. Shcherbakov in [12, 13]. In [12], superstable matrices are described, and the properties of linear systems with such matrices (superstable systems) are studied. While in [13], the problems of controller design, that makes the closed loop system superstable, are considered, and a static state regulator design in the form of solving the linear programming problem is also studied. In [14–17], superstability was used to solve other control theory problems.

The use of superstability in relation to switched systems is due to the following feature: any family of superstable systems of the same order forms an exponentially stable switched system for any switching law [18]. It is this property that was used in the mentioned articles [10, 11] to guarantee the stability when synthesizing a switched system. In [11], the theorem that extends the results of [13] to the case of switched systems is presented. In particular, [11, Theorem 3] provides the necessary and sufficient conditions for the existence of a regulator that makes the entire family of original systems superstable, but there is no constructive synthesis algorithm in [11].

In [13], it is shown that superstabilization by a static state regulator cannot always be achieved; therefore, the approach described in [11] may not be applicable since a superstabilizing regulator that simultaneously satisfies all the needed constraints does not exist. This article proposes a modification of the approach [11], in which, for non-superstabilizable subsystems, i.e. for the subsystems that cannot be made superstable by any static state regulator, one simply looks for a stabilizing regulator. The design procedure is reduced to the solution of a linear programming problem (as it was in [13]), in which linear constraints are divided into two types: the first one is given by systems that can be made superstable, and the second one by those which are not. For the latter systems, constraints giving the set of stabilizing regulators are determined using the D-decomposition technique [19, 20] by analyzing the stability of the characteristic polynomial of the closed loop system. The solution to the described linear programming problem only provides stability of individual subsystems of the switched system (i.e., it is a classical simultaneous stabilization problem), while the exponential stability property for an arbitrary switching law is only preserved for a subset of superstable matrices.

Another limitation of the approach is that it is applicable only to second order switched systems, since D-decomposition is difficult to implement for studying more than two parameters. However, one should note the general complexity of the problems of switched systems analysis and design, even in simple cases of low order [8, 21, 22]. In fact, the only effective method of analyzing and designing stable switched systems is to numerically verify the existence of a common quadratic Lyapunov function for all its subsystems [7, 23], which is a sufficient condition for the system stability for any switching signal.

The article is organized as follows. Section 2 describes superstable systems, their properties and their application in problems of linear switched systems stabilization. Section 3 contains a formal statement of the controller design problem. In Section 4, the problem is solved; the design algorithm is described, and the properties of the switched system closed by the found regulator are briefly analyzed. Section 5 provides a numerical example of the simultaneous stabilization problem solution using the proposed approach. The article ends with conclusions describing the main results and possible prospects for the approach improvement.

2. PRELIMINARIES

2.1. Linear Superstable Systems

Matrix $A \in \mathbb{R}^{n\times n}$ is called superstable [12], if it has on its main diagonal negative numbers that absolute values exceed the sum of off-diagonal elements absolute values in the same row. These conditions are presented as

$$-a_{ii} - \sum_{j\neq i} |a_{ij}| > 0, \quad i,j = \overline{1,n},$$

where a_{ij} is the element of the matrix A, placed in the ith row and jth column.

The value

$$\sigma(A) = \sigma = \min_i \left(-a_{ii} - \sum_{j\neq i} |a_{ij}| \right), \quad i,j = \overline{1,n}$$

is called *degree of superstability* of the matrix A.

A linear stationary system

$$\dot{x} = Ax, \quad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state vector, $x_0 \neq 0$ is the initial condition and the matrix A is superstable, is called *superstable system*.

It is shown in [12] that superstable system (1) is stable in the sense of $\mathrm{Re}\{\lambda(A)\} < 0$, where $\lambda(\cdot)$ are the matrix eigenvalues, and the system state for any initial condition is bounded by an estimate:

$$\|x\|_\infty \leqslant \|x_0\|_\infty e^{-\sigma t}, \quad t \geqslant 0, \tag{2}$$

where $\|x\|_\infty = \max_{1\leqslant i\leqslant n} |x_{[i]}|$ is the vector ∞-norm, $x_{[i]}$ is the ith component of the vector x. From this it follows, that superstable system (1) has non-quadratic Lyapunov function

$$V(x) = \|x\|_\infty. \tag{3}$$

This result turns out to be very useful for the analysis and design of switched systems.

2.2. Switched Systems: Superstability and Superstabilization

Let us consider the switched system

$$\dot{x}(t) = A_{\rho(t)}x(t), \quad x(0) = x_0 \neq 0, \tag{4}$$

where $A_{\rho(t)} \in \mathcal{A} = \{A_s \in \mathbb{R}^{n\times n} : \ s \in \mathcal{S}\}$ is the active (operational) matrix in the current time instance, $\mathcal{S} = \{1,\dots,S\}$, $S \in \mathbb{N}$ is the finite set of indices. The matrix $A_{\rho(t)}$ value is determined by piecewise constant measurable switching signal

$$\rho(t) : [0,\infty) \to \mathcal{S}. \tag{5}$$

Thus, $A_{\rho(\cdot)}$ is a matrix-valued function on the real axis, taking values from the set of real matrices $\mathcal{A}$.

Some additional limitations on $\rho(t)$ will be discussed separately in the corresponding sections. Further, the dependence on time of the function $\rho(t)$ is omitted in indices for simplicity.

It is proved in [18, Theorem 1], that if the set $\mathcal{A}$ contains only superstable matrices, then system (4) is exponentially stable for any switching law (5) and arbitrary initial conditions x_0. Such a system is usually called a superstable switched system. Moreover, this result holds for any (including infinite) sets of superstable matrices of any size.

The proof of the corresponding theorem in [18] is based on the inequality (2) and the analysis of the state vector upper bounds for system (4). More simply, [18, Theorem 1] is a corollary of (3), since $\|x\|_\infty$ is the common Lyapunov function for all superstable systems.

The following dynamic switched system was considered in [10, 11]:

$$\dot{x} = A_\rho x + b_\rho u, \quad A_\rho \in \mathcal{A}, \quad x \in \mathbb{R}^n, \quad b_\rho \in \mathcal{B} = \{b_s \in \mathbb{R}^n, \ s \in \mathcal{S}\}, \tag{6}$$

where b_ρ is the active vector multiplier for control, that is introduced in the same way as system matrix A_ρ in (4), $u \in \mathbb{R}$ is the control signal formed by the linear static state regulator:

$$u = k^\top x,$$

where $k \in \mathbb{R}^n$ is the row-vector of its coefficients.

In [11] the authors considered the problem of finding a static regulator k that simultaneously makes superstable all matrices from the set

$$\mathcal{M} = \{M_s = A_s + b_s k^\top, \ s \in \mathcal{S}\} \tag{7}$$

of the closed loop system $\dot{x}(t) = M_\rho x(t) = (A_\rho + b_\rho k^\top)x(t)$, $M_\rho \in \mathcal{M}$. If such a regulator exists, then it is called *simultaneously superstabilizing*.

In (7) and further, the subscript "s" is used for matrices (and their elements) to emphasize that these notations refer to elements of the sets $\mathcal{A}$ and $\mathcal{M}$, and not to the active matrices A_ρ or M_ρ.

In [11, Theorem 3] the necessary and sufficient conditions for the existence of a simultaneously superstabilizing regulator are formulated. They are determined by the solution feasibility of a special system of linear inequalities and are verified numerically.

From a practical point of view, it is convenient to represent these conditions in the form of the following linear programming problem:

$$\begin{gathered} \max \sigma, \\ -(a_{ij}^s + b_i^s k_j) - \sum_{j \neq i} n_{ij}^s \geqslant \sigma, \quad i = \overline{1,n}, \\ -n_{ij}^s \leqslant a_{ij}^s + b_i^s k_j \leqslant n_{ij}^s, \quad i,j = \overline{1,n}, \ i \neq j, \ s \in \mathcal{S}, \end{gathered} \tag{8}$$

where $a_{ij}^s + b_i^s k_j = m_{ij}^s$ are the elements of the matrices from set (7) of the closed loop system. In problem (8) the variables are: the degree of superstability σ, the regulator coefficients k_j and non-negative auxiliary scalar variables n_{ij}^s.

In this formulation, the conditions for the existence of a simultaneously superstabilizing regulator (or the solvability of problem (8)) are given by Theorem 2.1 from [13], which states, that if σ, k are the solution to problem (8) and $\sigma > 0$, then the regulator k is the simultaneously superstabilizing one. In the opposite case (when $\sigma \leqslant 0$) the system cannot be made superstable by the control $u = k^\top x$.

3. PROBLEM STATEMENT

Let us consider the controller design problem of a superstabilizing static state regulator for a second order switched system of the form

$$\dot{x} = A_\rho x + bu, \ u = k^\top x, \quad A_\rho \in \mathcal{A} = \{A_s \in \mathbb{R}^{2\times 2}, \ s \in \mathcal{S}\}, \tag{9}$$

where, in contrast to (6), the matrices A_s of the set $\mathcal{A}$ have dimension 2, and the vector b is given and is not subject to switching.

It is not always possible to design a simultaneously superstable regulator by solving problem (8), even in this narrower and simpler case. In [20, p. 138] the authors show several cases for fixed matrices in (9), for which superstabilization is a priori impossible: for example, when the vector $b \in \mathbb{R}^2$ contains zero elements, and the corresponding row of the matrix A does not satisfy the conditions of superstability.

Further in (9) we set $b = [1\ 1]^\top$. For this case, in [13] a necessary and sufficient condition for superstabilizability of the separate matrices A_s is obtained:

$$\tau(A_s) = a^s_{11} - a^s_{21} + a^s_{22} - a^s_{12} < 0; \tag{10}$$

and if $\tau(A_s) \geqslant 0,\ s \in \mathcal{S}$, then superstabilization by a static state regulator (for the certain matrix A_s) is impossible.

It is also assumed that among the matrices $A_s \in \mathcal{A}$ there are those that cannot be made superstable according to condition (10). A family of such matrices is denoted by $A_\phi,\ \phi \in \mathcal{S}_\phi \subset \mathcal{S}$. The remaining matrices, that can be made superstable, are denoted as $A_\psi,\ \psi \in \mathcal{S}_\psi \subset \mathcal{S}$, naturally $\mathcal{S}_\psi \cup \mathcal{S}_\phi = \mathcal{S}$.

Let us consider the following design problem.

Problem 1. For system (9), find a static state stabilizing regulator, which for the family of matrices A_ψ provides the superstability of the matrices $M_\psi = A_\psi + bk^\top$ of the closed loop system, and for the remaining matrices A_ϕ just provides stability of the closed loop system.

The Problem 1 is a simultaneous stabilization problem, when a common stabilizing regulator is sought for several plants (subsystems), and it is impossible to guarantee its solvability in advance. In this case, stability of the switched system closed by the found regulator is also not guaranteed (unlike Problem (8)). The properties of the solution to the Problem 1 in relation to the switched systems stability are described in more detail in the next section.

4. STABILIZING CONTROLLER DESIGN

Due to the presence of matrices A_ϕ, $\phi \in \mathcal{S}_\phi$ that cannot be made superstable, solving the Problem 1 using (8) is obviously impossible.

To solve Problem 1, an approach based on solving the linear programming problem is proposed, that can also be treated as a modification of Problem (8), in which for the matrices, that cannot be made superstable, constraints linear in k are used to ensure stability of the closed loop system with the matrices M_ϕ. Such constraints can be obtained using the D-decomposition technique [19, 20], which is described below.

For simplicity, the indices ϕ of the matrices A and M and their elements are omitted. Considering $b = [1\ 1]^\top$, a characteristic polynomial of the closed loop system matrix

$$M = A + bk^\top = \begin{bmatrix} a_{11} + k_1 & a_{12} + k_2 \\ a_{21} + k_1 & a_{22} + k_2 \end{bmatrix}$$

takes the form

$$d(s, k) = \det(sI - M) = s^2 - s(k_1 + k_2 + a_{11} + a_{22}) + k_1(a_{22} - a_{12}) + k_2(a_{11} - a_{21}) + \det(A).$$

According to [20], when k changes, the location of the $d(s, k)$ roots can change only if the real root (or the real part of complex conjugate roots pair) crosses zero (crosses the imaginary axis on the complex plane). The corresponding boundaries of the D-decomposition are described by the parametric equation $d(j\omega, k) = 0$ or in explicit form:

$$\begin{cases} -\omega^2 + k_1(a_{22} - a_{12}) + k_2(a_{11} - a_{21}) + \det(A) = 0, \\ \omega(k_1 + k_2 + a_{11} + a_{22}) = 0. \end{cases} \tag{11}$$

For $\omega = 0$ system (11) has a singular solution

$$k_1(a_{22} - a_{12}) + k_2(a_{11} - a_{21}) + \det(A) = 0, \tag{12}$$

which gives us the equation of one of the boundaries.

The second boundary is obtained from the second equation in (11):

$$k_1 + k_2 + a_{11} + a_{22} = 0, \tag{13}$$

with the difference that the boundary is not the entire line, but a ray outgoing from the point of intersection with the line (12), which is parametrically determined by the following functions:

$$\begin{gathered} k_1(\omega) = \frac{1}{r}(\omega^2 + a_{11}^2 - a_{21}[a_{11} + a_{22} - a_{12}]), \\ k_2(\omega) = -\frac{1}{r}(\omega^2 + a_{22}^2 - a_{12}[a_{11} + a_{22} - a_{21}]), \\ r = -a_{11} - a_{12} + a_{21} + a_{22}, \quad \omega \in (0, \infty). \end{gathered} \tag{14}$$

The boundaries of the D-decomposition, determined by (12) and (13), (14), split the regulator parameters plane (k_1, k_2) into the three areas:

—the real parts of both roots of the polynomial $d(s, k)$ are non-negative ($\mathrm{Re}\{\lambda_{1,2}(M)\} \geqslant 0$);

—one root of the polynomial $d(s, k)$ is stable, but the second one is not ($\mathrm{Re}\{\lambda_1(M)\} < 0$, $\mathrm{Re}\{\lambda_2(M)\} \geqslant 0$);

—both roots of the polynomial $d(s, k)$ are stable ($\mathrm{Re}\{\lambda_{1,2}(M)\} < 0$).

The last area, where the closed loop system matrix M is stable, defines a set of stabilizing regulators for the corresponding matrix A. The boundaries of this set are determined by linear constraints obtained from (12), (13).

Using the determined constraints, the following linear programming problem that solves Problem 1 is formulated:

$$\begin{gathered} \max \sigma, \\ -(a_{ij}^{\psi} + k_j) - \sum_{j \neq i} n_{ij}^{\psi} \geqslant \sigma, \quad i = 1, 2, \\ -n_{ij}^{\psi} \leqslant a_{ij}^{\psi} + k_j \leqslant n_{ij}^{\psi}, \quad i, j = 1, 2,\ i \neq j,\ \psi \in \mathcal{S}_{\psi}, \\ c_i^{\phi} k^{\top} + z_i^{\phi} < 0, \quad i = 1, 2,\ \phi \in \mathcal{S}_{\phi}. \end{gathered} \tag{15}$$

In (15), constraints on the elements of closed subsystems matrices M_{ψ}, $\psi \in \mathcal{S}_{\psi}$ coincide with the similar ones in (8). Vectors $c_i^{\phi} \in \mathbb{R}^2$ and scalars $z_i^{\phi} \in \mathbb{R}$, found using D-decomposition, define linear constraints guaranteeing stability of the closed loop system for matrices M_{ϕ}.

Now, an algorithm for solving Problem 1 for system (9) can be formulated.

Algorithm 1.

1. Find subsets A_{ϕ} and A_{ψ} of the plant matrices $A_s \in \mathcal{A}$ using (10).

2. Carry out D-decomposition for matrices A_{ψ} according to the described approach and find the constraint parameters using (12), (13).

3. Create and solve linear programming problem (15).

The existence of a solution to problem (15) is determined by the same conditions as for problem (8): if a solution k, $\sigma > 0$ is found, then Problem 1 has been solved.

The described algorithm can be used to solve a general problem with an arbitrary vector $b = [b_1\ b_2]^\top \in \mathbb{R}^2$ in the system (9). However, in this case, criterion (10) cannot be used to determine the set A_ϕ, $\phi \in \mathcal{S}_\phi$. So, in the first step of Algorithm 1 the superstabilizability of the matrices A_s, $s \in \mathcal{S}$ should be checked using the design procedure from [13, section 2]. System (11) will respectively take the form

$$\begin{cases} -\omega^2 + k_1(b_1 a_{22} - b_2 a_{12}) + k_2(b_2 a_{11} - b_1 a_{21}) + \det(A) = 0, \\ \omega(k_1 b_1 + k_2 b_2 + a_{11} + a_{22}) = 0. \end{cases}$$

The computational formulae (12), (13) and (14) for finding the boundaries of the stability area will change accordingly.

It should be noted that the first step of Algorithm 1 allows one to check only the necessary condition for simultaneous superstabilization, namely, the superstabilizability of individual matrices (moreover for the fixed vector b). However, even if all matrices A_s, $s \in \mathcal{S}$ can individually be made superstable, it may occur that together they are not simultaneously superstabilizable. In this case, Algorithm 1 can be slightly modified: one can form the family A_ψ "manually", using the following condition as a goal criterion

$$\max_{\mathcal{S}} |\mathcal{S}_\psi|.$$

i.e. try to find a solvable problem (15) in which the maximum possible number of matrices from $\mathcal{A}$ can simultaneously be made superstable. Such a problem is combinatorial and for small dimensions of $\mathcal{S}$ can be solved by simple brute-force search. For larger dimensions of $\mathcal{S}$, one can form the set $\mathcal{S}_\psi$ by analysis of the intersection area of sets of matrices A_s, $s \in \mathcal{S}$ superstabilizing regulators, which can be constructed explicitly by studying the inequalities $-m_{ii}^s > |m_{ij}^s|$, $i, j = 1, 2$, $i \neq j$, for more details see the example in Section 5.

Note also, that there is a singular case when no matrix in $\mathcal{A}$ can be made superstable using a static regulator. Then the usage of the proposed approach is reduced to analyzing the intersection area of sets of the stabilizing regulators, which are found by D-decomposition. If the intersection area is non-empty, then it constitutes a set of simultaneously stabilizing state regulators. Otherwise, if it is empty, then such regulators do not exist. Other cost criteria of the regulators can be optimized over the found set [23, 24]. Herewith, the set of simultaneously stabilizing regulators has a simple form, since it is determined by a system of linear inequalities.

It was mentioned above that switched system (9) designed using (15) can lose the property of exponential stability for any switching signal, which is provided by the solution of problem (8). More precisely, it is preserved only for switching between superstabilizable matrices A_ψ, $\psi \in \mathcal{S}_\psi$, while for whole system (9) only the necessary condition of the switched system stability is satisfied: the stability of each individual matrix in the set $\mathcal{M}$.

Due to the last property, it is possible to use known approaches to providing stability of system (9) by adding certain restrictions on the switching law $\rho(t)$. For example, there is a known approach in which the maximum switching rate is limited, which is implemented using the concept of *dwell-time* denoted by $t_d > 0$. It is introduced as the minimum time interval between two adjacent switchings, that guarantees the switched system stability for any switching sequence. In [25], an algorithm for finding the upper bound of t_d is proposed, which is based on the following theorem.

Theorem 1 [25]. *Let $\widehat{t}_d > 0$ be a given number. If there exists such $P_i \in \mathbb{R}^{2\times 2}$, $i \in \mathcal{S}$, that*

$$\begin{cases} P_i > 0, \quad \forall i, \\ A_i^\top P_i + P_i A_i < 0, \quad \forall i, \\ e^{A_i^\top \widehat{t}_d} P_j e^{A_i \widehat{t}_d} < P_i, \quad \forall i \neq j, \end{cases}$$

then system (9) is exponentially stable for any arbitrary switching law $\rho(t)$, constrained by the dwell-time $\widehat{t}_d$.

The estimate of t_d is determined by a one-dimensional search: first, some sufficiently large number $\tau_0 > 0$ is selected for which the conditions of Theorem 1 at $A_i = M_i$, $\forall i$ are satisfied. Then for the sought estimate the following relation $\widehat{t}_d \in (0,\ \tau_0]$ holds. Next, $\widehat{t}_d$ is determined with the desired accuracy by the binary search method. The found dwell-time estimate is the best in the sense of quadratic Lyapunov functions.

5. EXAMPLE

Let us consider system (9) with $b = [1\ 1]^\top$ and with the set of matrices $\mathcal{A}$:

$$A_1 = \begin{bmatrix} -1 & 1 \\ -3 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad \mathcal{S} = \{1,\ 2,\ 3\}, \tag{16}$$

where A_s have the following values of $\tau(A_s)$ according to (10) and eigenvalues $\lambda(A_s)$:

$$\begin{array}{lll} \tau(A_1) = -1, & \tau(A_2) = -5, & \tau(A_3) = 1, \\ \lambda(A_1) = -1.5 \pm j1.658, & \lambda(A_2) = \{3.372,\ -2.372\}, & \lambda(A_3) = 1 \pm j1.414. \end{array}$$

Matrices A_1 and A_2 can be made superstable, but matrix A_3 cannot; according to the first step of Algorithm 1, the sets of matrices $A_\psi = \{A_1,\ A_2\}$ and $A_\phi = A_3$ are formed.

Using the D-decomposition technique based on (12)–(14), the set of stabilizing regulators for the matrix A_3 is determined, its boundaries are defined by the following inequalities:

$$\begin{gathered} 2k_1 - k_2 + 3 > 0, \\ k_1 + k_2 + 2 < 0. \end{gathered}$$

Proceeding to the constraints form used in (15), one gets:

$$\begin{aligned} c_1 &= [-2\ 1]^\top, \quad z_1 = -3, \\ c_2 &= [1\ 1]^\top, \quad z_2 = 2, \end{aligned}$$

where for simplicity, the superscript ϕ is omitted, since A_ϕ consists of only one matrix.

Figure 1a shows how the described technique is used to analyze matrix A_3: in the space of regulator coefficients $(k_1,\ k_2)$, the solid straight line is defined by $2k_1 - k_2 + 3 = 0$, and the dotted straight line is defined by $k_1 + k_2 + 2 = 0$. Its visible part is not the boundary; i.e., the number 2 denotes the entire halfplane "to the left" of the solid line.

The sought area of the stabilizing regulators is shaded, and numbers 1 and 2 correspond to the remaining areas in the order according to (14): in the first one, both eigenvalues of the closed loop system matrix are unstable $((\mathrm{Re}\{\lambda_{1,2}(M_3)\} > 0)$, and in the second one, only one of them is unstable $(\mathrm{Re}\{\lambda_1(M_3)\} < 0,\ \mathrm{Re}\{\lambda_2(M_3)\} > 0)$.

At the last step of solving Problem 1, linear programming problem (15) is formed and numerically solved using appropriate software. In this article, MATLAB with the cvx package was used.

As a result of the design, the following regulator was found:

$$k^\top = [-2.635\ \ -4.135], \tag{17}$$

which provides that the matrices M_1 and M_2 are superstable with the degree of superstability $\sigma = 0.5 > 0$, and the matrix M_3 is stable. Consequently, Problem 1 of simultaneous stabilization has been solved.

In Fig. 1b the design procedure is illustrated: solid lines demonstrate the boundaries of sets of the superstabilizing regulators for matrices A_1 and A_2, obtained from the analysis of inequalities

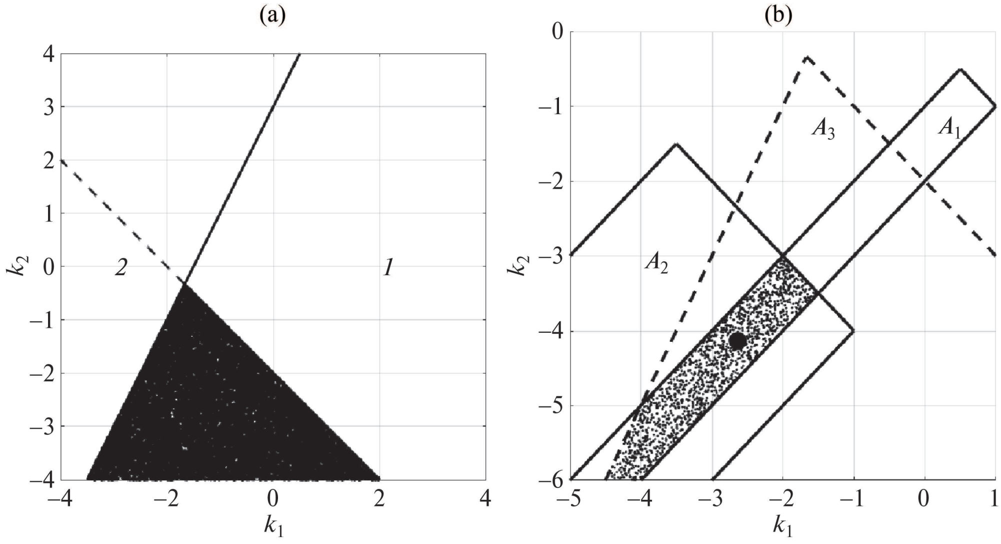

Fig. 1. (a) The boundaries of D-decomposition for matrix A_3; (b) the results of problem (15) solution: the designed regulator is shown by the large black dot.

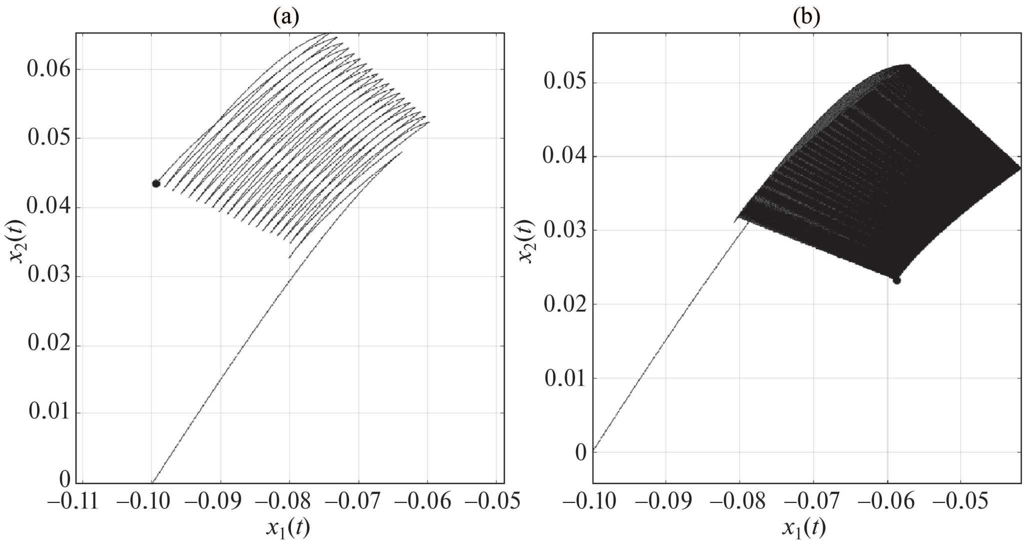

Fig. 2. The phase portrait of system (9), (16), (17) at $x(0) = [-0{,}1\ 0]^\top$ and the dwell-times: (a) $t_d = 0.2$ s and (b) $t_d = 0.255$ s.

$-m_{ii}^{\psi} > |m_{ij}^{\psi}|$, $i, j = 1, 2$, $i \neq j$, $\psi = 1, 2$, and the dotted line shows the boundaries of set of the stabilizing regulators for matrix A_3. The large black dot denotes the solution found numerically, while the small dots around it highlight the area containing regulators that satisfy the constraints in the corresponding problem (15).

System (9) with matrices (16) and regulator (17) is not exponentially stable for arbitrary switching law $\rho(t)$. Figure 2a shows the trajectory of system (9), (16), (17) at $x(0) = [-0.1\ 0]^\top$ and the signal $\rho(t)$, which changes the active matrix after a time interval $t_d = 0.2$ s, and switchings are made only between matrices A_1 and A_3 (or, respectively, only between M_1 and M_3 in the closed loop system). In this case, the trajectory of the system is not bounded ($\|x(t)\|_\infty \to \infty$ as $t \to \infty$).

Using the described algorithm [25], an estimate of the dwell-time $\widehat{t}_d = 0.254$ s was found. In Fig. 2b demonstrates the phase portrait of the system under the same conditions that were used for Fig. 2a, but with a dwell-time $t_d = 0.255$ s, from which it is clear that the system trajectory converges to zero over time.

Note that a numerical verification of the approach to simultaneous stabilization described in [7, 20] showed that for matrices $\mathcal{A}$ if a static state regulator is designed, the corresponding linear matrix inequality, that guarantees existence of a common quadratic Lyapunov function, is infeasible.

6. CONCLUSION

The article proposes an approach to solving the simultaneous stabilization problem of second order linear systems by a static state regulator, which is the modification of the well-known method for designing switched systems based on superstability. The controller design procedure, as it happens in the original method, is reduced to solving a special linear programming problem. The controller design algorithm and the brief analysis of the designed regulator properties in the stability sense of the switched system are presented. The proposed design procedure is illustrated with a numerical example.

The new procedure extends the class of solvable problems to matrices that cannot be made superstable, at the cost of losing the ensured exponential stability of the switched system. It is preserved only for a subset of superstable subsystems. Partial compensation of this drawback is carried out by limiting the switching rate, which is implemented by estimating the minimum dwell-time using the known method.

Development of this approach includes an attempt to take into account dwell-time in the design procedure. As an extreme case, one can discard superstability altogether and formulate the problem of searching for the dwell-time optimal static regulator over a set of simultaneously stabilizing regulators found using the described D-decomposition procedure.

Potential problem for further research is the analysis of classes "analogous" to superstability, which also a priori guarantee the existence of a common Lyapunov function for matrices of the switched system, but are broader. The most obvious candidate in this direction seems to be diagonally stable matrices, a subset of which are superstable ones.

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